**Matroid maps**

A.V. Borovik, Department of Mathematics, UMIST

**1. Notation**

This paper continues the works [1,2] and uses, with some modification, their terminology and notation. Throughout the paper W is a Coxeter group (possibly infinite) and P a finite standard parabolic subgroup of W. We identify the Coxeter group W with its Coxeter complex and refer to elements of W as chambers, to cosets with respect to a parabolic subgroup as residues, etc. We shall use the calligraphic letter as a notation for the Coxeter complex of W and the symbol for the set of left cosets of the parabolic subgroup P. We shall use the Bruhat ordering on in its geometric interpretation, as defined in [2, Theorem 5.7]. The w-Bruhat ordering on is denoted by the same symbol as the w-Bruhat ordering on . Notation , <w, >w has obvious meaning.

We refer to Tits [6] or Ronan [5] for definitions of chamber systems, galleries, geodesic galleries, residues, panels, walls, half-complexes. A short review of these concepts can be also found in [1,2].

**2. Coxeter matroids**

If W is a finite Coxeter group, a subset is called a Coxeter matroid (for W and P) if it satisfies the maximality property: for every the set contains a unique w-maximal element A; this means that for all . If is a Coxeter matroid we shall refer to its elements as bases. Ordinary matroids constitute a special case of Coxeter matroids, for W=Symn and P the stabiliser in W of the set [4]. The maximality property in this case is nothing else but the well-known optimal property of matroids first discovered by Gale [3].

In the case of infinite groups W we need to slightly modify the definition. In this situation the primary notion is that of a matroid map

i.e. a map satisfying the matroid inequality

The image of obviously satisfies the maximality property. Notice that, given a set with the maximality property, we can introduce the map by setting be equal to the w-maximal element of . Obviously, is a matroid map. In infinite Coxeter groups the image of the matroid map associated with a set satisfying the maximality property may happen to be a proper subset of (the set of all `extreme' or `corner' chambers of ; for example, take for a large rectangular block of chambers in the affine Coxeter group ). This never happens, however, in finite Coxeter groups, where .

So we shall call a subset a matroid if satisfies the maximality property and every element of is w-maximal in with respect to some . After that we have a natural one-to-one correspondence between matroid maps and matroid sets.

We can assign to every Coxeter matroid for W and P the Coxeter matroid for W and 1 (or W-matroid).

Теорема 1. [2, Lemma 5.15] A map

is a matroid map if and only if the map

defined by is also a matroid map.

Recall that denotes the w-maximal element in the residue . Its existence, under the assumption that the parabolic subgroup P is finite, is shown in [2, Lemma 5.14].

In is a matroid map, the map is called the underlying flag matroid map for and its image the underlying flag matroid for the Coxeter matroid . If the group W is finite then every chamber x of every residue is w-maximal in for w the opposite to x chamber of and , as a subset of the group W, is simply the union of left cosets of P belonging to .

**3. Characterisation of matroid maps**

Two subsets A and B in are called adjacent if there are two adjacent chambers and , the common panel of a and b being called a common panel of A and B.

Лемма 1. If A and B are two adjacent convex subsets of then all their common panels belong to the same wall .

We say in this situation that is the common wall of A and B.

For further development of our theory we need some structural results on Coxeter matroids.

Теорема 2. A map is a matroid map if and only if the following two conditions are satisfied.

(1) All the fibres , , are convex subsets of .

(2) If two fibres and of are adjacent then their images A and B are symmetric with respect to the wall containing the common panels of and , and the residues A and B lie on the opposite sides of the wall from the sets , , correspondingly.

Доказательство. If is a matroid map then the satisfaction of conditions (1) and (2) is the main result of [2].

Assume now that satisfies the conditions (1) and (2).

First we introduce, for any two adjacent fibbers and of the map , the wall separating them. Let be the set of all walls .

Now take two arbitrary residues and chambers and . We wish to prove .

Consider a geodesic gallery

connecting the chambers u and v. Let now the chamber x moves along from u to v, then the corresponding residue moves from to . Since the geodesic gallery intersects every wall no more than once [5, Lemma 2.5], the chamber x crosses each wall in no more than once and, if it crosses , it moves from the same side of as u to the opposite side. But, by the assumptions of the theorem, this means that the residue crosses each wall no more than once and moves from the side of opposite u to the side containing u. But, by the geometric interpretation of the Bruhat order, this means [2, Theorem 5.7] that decreases, with respect to the u-Bruhat order, at every such step, and we ultimately obtain

**Список литературы**

Borovik A.V., Gelfand I.M. WP-matroids and thin Schubert cells on Tits systems // Advances Math. 1994. V.103. N.1. P.162-179.

Borovik A.V., Roberts K.S. Coxeter groups and matroids, in Groups of Lie Type and Geometries, W. M. Kantor and L. Di Martino, eds. Cambridge University Press. Cambridge, 1995 (London Math. Soc. Lect. Notes Ser. V.207) P.13-34.

Gale D., Optimal assignments in an ordered set: an application of matroid theory // J. Combinatorial Theory. 1968. V.4. P.1073-1082.

Gelfand I.M., Serganova V.V. Combinatorial geometries and torus strata on homogeneous compact manifolds // Russian Math. Surveys. 1987. V.42. P.133-168.

Ronan M. Lectures on Buildings - Academic Press. Boston. 1989.

Tits J. A local approach to buildings, in The Geometric Vein (Coxeter Festschrift) Springer-Verlag, New York a.o., 1981. P.317-322.