**Matroid maps**

A.V. Borovik, Department of Mathematics, UMIST

**1. Notation**

This paper continues the works [1,2] and uses, with some modification, their terminology and notation. Throughout the paper W is a Coxeter group (possibly infinite) and P a finite standard parabolic subgroup of W. We identify the Coxeter group W with its Coxeter complex and refer to elements of W as chambers, to cosets with respect to a parabolic subgroup as residues, etc. We shall use the calligraphic letter as a notation for the Coxeter complex of W and the symbol for the set of left cosets of the parabolic subgroup P. We shall use the Bruhat ordering on in its geometric interpretation, as defined in [2, Theorem 5.7]. The w-Bruhat ordering on is denoted by the same symbol as the w-Bruhat ordering on . Notation , <w, >w has obvious meaning.



We refer to Tits [6] or Ronan [5] for definitions of chamber systems, galleries, geodesic galleries, residues, panels, walls, half-complexes. A short review of these concepts can be also found in [1,2].

**2. Coxeter matroids**

If W is a finite Coxeter group, a subset is called a Coxeter matroid (for W and P) if it satisfies the maximality property: for every the set contains a unique w-maximal element A; this means that for all . If is a Coxeter matroid we shall refer to its elements as bases. Ordinary matroids constitute a special case of Coxeter matroids, for W=Symn and P the stabiliser in W of the set [4]. The maximality property in this case is nothing else but the well-known optimal property of matroids first discovered by Gale [3].



In the case of infinite groups W we need to slightly modify the definition. In this situation the primary notion is that of a matroid map



i.e. a map satisfying the matroid inequality



The image of obviously satisfies the maximality property. Notice that, given a set with the maximality property, we can introduce the map by setting be equal to the w-maximal element of . Obviously, is a matroid map. In infinite Coxeter groups the image of the matroid map associated with a set satisfying the maximality property may happen to be a proper subset of (the set of all `extreme' or `corner' chambers of ; for example, take for a large rectangular block of chambers in the affine Coxeter group ). This never happens, however, in finite Coxeter groups, where .



So we shall call a subset a matroid if satisfies the maximality property and every element of is w-maximal in with respect to some . After that we have a natural one-to-one correspondence between matroid maps and matroid sets.



We can assign to every Coxeter matroid for W and P the Coxeter matroid for W and 1 (or W-matroid).



Теорема 1. [2, Lemma 5.15] A map



is a matroid map if and only if the map



defined by is also a matroid map.



Recall that denotes the w-maximal element in the residue . Its existence, under the assumption that the parabolic subgroup P is finite, is shown in [2, Lemma 5.14].



In is a matroid map, the map is called the underlying flag matroid map for and its image the underlying flag matroid for the Coxeter matroid . If the group W is finite then every chamber x of every residue is w-maximal in for w the opposite to x chamber of and , as a subset of the group W, is simply the union of left cosets of P belonging to .



**3. Characterisation of matroid maps**

Two subsets A and B in are called adjacent if there are two adjacent chambers and , the common panel of a and b being called a common panel of A and B.



Лемма 1. If A and B are two adjacent convex subsets of then all their common panels belong to the same wall .



We say in this situation that is the common wall of A and B.



For further development of our theory we need some structural results on Coxeter matroids.

Теорема 2. A map is a matroid map if and only if the following two conditions are satisfied.



(1) All the fibres , , are convex subsets of .



(2) If two fibres and of are adjacent then their images A and B are symmetric with respect to the wall containing the common panels of and , and the residues A and B lie on the opposite sides of the wall from the sets , , correspondingly.



Доказательство. If is a matroid map then the satisfaction of conditions (1) and (2) is the main result of [2].



Assume now that satisfies the conditions (1) and (2).



First we introduce, for any two adjacent fibbers and of the map , the wall separating them. Let be the set of all walls .



Now take two arbitrary residues and chambers and . We wish to prove .



Consider a geodesic gallery



connecting the chambers u and v. Let now the chamber x moves along from u to v, then the corresponding residue moves from to . Since the geodesic gallery intersects every wall no more than once [5, Lemma 2.5], the chamber x crosses each wall in no more than once and, if it crosses , it moves from the same side of as u to the opposite side. But, by the assumptions of the theorem, this means that the residue crosses each wall no more than once and moves from the side of opposite u to the side containing u. But, by the geometric interpretation of the Bruhat order, this means [2, Theorem 5.7] that decreases, with respect to the u-Bruhat order, at every such step, and we ultimately obtain



**Список литературы**

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