**On a decomposition of an element of a free metabelian group as a productof primitive elements**

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**1. Introduction**

Let G=Fn/V be a free in some variety group of rank n. An element is called primitive if and only if g can be included in some basis g=g1,g2,...,gn of G. The aim of this note is to consider a presentation of elements of free groups in abelian and metabelian varieties as a product of primitive elements. A primitive length |g|pr of an element is by definition a smallest number m such that g can be presented as a product of m primitive elements. A primitive length |G|pr of a group G is defined as , so one can say about finite or infinite primitive length of given relatively free group.



Note that |g|pr is invariant under action of Aut G. Thus this notion can be useful for solving of the automorphism problem for G.

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**2. Presentation of elements of a free abelian group of rank n as a product of primitive elements**

Let An be a free abelian group of rank n with a basis a1,a2,...,an. Any element can be uniquelly written in the form



.



Every such element is in one to one correspondence with a vector . Recall that a vector (k1,...,kn) is called unimodular, if g.c.m.(k1,...,kn)=1.



Лемма 1. An element of a free abelian group An is primitive if and only if the vector (k1,...,kn) is unimodular.



Доказательство. Let , then . If c is primitive, then it can be included into a basis c=c1,c2,...,cn of the group An. The group (n factors) in such case, has a basis , where means the image of ci. However, , that contradics to the well-known fact: An(d) is not allowed generating elements. Conversely, it is well-known , that every element c=a1k1,...,ankn such that g.c.m.(k1,...,kn)=1 can be included into some basis of a group An.



Note that every non unimodular vector can be presented as a sum of two unimodular vectors. One of such possibilities is given by formula (k1,...,kn)=(k1-1,1,k3,...,kn)+(1,k2-1,0,...,0).



Предложение 1. Every element , , can be presented as a product of not more then two primitive elements.



Доказательсво. Let c=a1k1...ankn for some basis a1,...an of An. If g.c.m.(k1,...,kn)=1, then c is primitive by Lemma 1. If , then we have the decomposition (k1,...,kn)=(s1,...,sn)+(t1,...,tn) of two unimodular vectors. Then c=(a1s1...ansn)(a1t1...antn) is a product of two primitive elements.



Corollary.It follows that |An|pr=2 for . ( Note that .



**3. Decomposition of elements of the derived subgroup of a free metabelian group of rank 2 as a product of primitive ones**

Let be a free metabelian group of rank 2. The derived subgroup M'2 is abelian normal subgroup in M2. The group is a free abelian group of rank 2. The derived subgroup M'2 can be considered as a module over the ring of Laurent polynomials



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The action in the module M'2 is determined as ,where is any preimage of element in M2, and



.



Note that for , we have



(u,g)=ugu-1g-1=u1-g.

Any automorphism is uniquelly determined by a map



.



Since M'2 is a characteristic subgroup, induces automorphism of the group A2 such that



Consider an automorphism of the group M2, identical modM'2, which is defined by a map



,



By a Bachmuth's theorem from [1] is inner, thus for some we have



Consider a primitive element of the form ux, . By the definition there exists an automorphism such that



|  |  |
| --- | --- |
|  | (1) |



Using elementary transformations we can find a IA-automorphism with a first row of the form(1). Then by mentioned above Bachmuth's theorem



In particular the elements of type u1-xx, u1-yy, are primitive.



Предложение 2. Every element of the derived subgroup of a free metabelian group M2 can be presented as a product of not more then three primitive elements.

Доказательство. Every element can be written as , and can be presented as



.



Thus,



|  |  |
| --- | --- |
|  | (2) |

A commutator , by well-known commutator identities can be presented as



|  |  |
| --- | --- |
|  | (3) |

The last commutator in (3) can be added to first one in (2). We get [y-1 , that is a product of three primitive elements.



**4. A decomposition of an element of a free metabelian group of rank 2 as a product of primitive elements**

For further reasonings we need the following fact: any primitive element of a group A2 is induced by a primitive element , . It can be explained in such way. One can go from the basis to some other basis by using a sequence of elementary transformations, which are in accordance with elementary transformations of the basis <x,y> of the group M2.



The similar assertions are valid for any rank .



Предложение 3. Any element of group M2 can be presented as a product of not more then four primitive elements.

Доказательство. At first consider the elements in form . An element is primitive in A2 by lemma 1, consequently there is a primitive element of type . Hence, Since, an element is primitive, it can be included into some basis inducing the same basis of A2. After rewriting in this new basis we have:



,



and so as before



Obviously, two first elements above are primitive. Denote them as p1, p2. Finally, we have

, a product of three primitive elements.



If , then by proposition 1 we can find an expansion as a product of two primitive elements, which correspond to primitive elements of M2: v1xk1yl1,v2xk2yl2,v1,v2 .



Further we have the expansion



The element w(v1xk1yl1) can be presented as a product of not more then three primitive elements. We have a product of not more then four primitive elements in the general case.

5. A decomposition of elements of a free metabelian group of rank as a product of primitive elements



Consider a free metabelian group Mn=<x1,...,xn> of rank .



Предложение 4. Any element can be presented as a product of not more then four primitive elements.



Доказательсво. It is well-known [2], that M'n as a module is generated by all commutators . Therefore, for any there exists a presentation



Separate the commutators from (4) into three groups in the next way.

1) - the commutators not including the element x2 but including x1.



2) - the other commutators not including the x1.



3) And the third set consists of the commutator .



Consider an automorphism of Mn, defining by the following map:

,



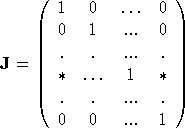
.



The map determines automorphism, since the Jacobian has a form



,



and hence, det Jk=1.

Since element can be included into a basis of Mn, it is primitive. Thus any element can be presented in form



x3x2x1]



[x1-1x2-1x3-1]. =p1p2p3p4 a product of four primitive elements.

Note that the last primitive element p4=x1-1x2-1x3-1 can be arbitrary.

Предложение 5. Any element of a free metabelian group Mn can be presented as a product of not more then four primitive elements.

Доказательство. Case 1. Consider an element , so that g.c.m.(k1,...,kn)=1. An element is primitive by lemma 1 and there exists a primitive element ,



An element from derived subgroup can be presented as a product of not more then four primitive elements with a fixed one of them:



Then .



Case 2. If , then by lemma 2 , where are primitive in An. There exist primitive elements So We have just proved that the element wp1 can be presented as a product of not more then three primitive elements p1'p2'p3'. Finally we have c=p1'p2'p3'p2, a product of not more then four primitive elements.



**Список литературы**

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